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SCATTERING OF SEISMIC BODY WAVES BY SMALL RANDOM INHOMOGENEITIES IN THE EARTH

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The theory supports the writer's earlier suggestion that observed precursors to the seismic core phase PKIKP may originate by scattering from the phases PKP_{1} and PKP_{2} by irregularities in the vicinity of the earth's mantle-core boundary.

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Summary

A theory of scattering of seismic body waves by small random spacial fluctuations in density and elastic parameters in an otherwise spherically symmetrical Earth model is developed. It is shown that a primary wave disturbance of either P or S type travelling through a slightly irregular solid medium will generate scattered waves of both P and S types. Explicit formulas are derived for the mean square amplitudes of P waves scattered from several different assumed forms of primary P wave.

The theory assumes that the primary wave may be locally approximated by a plane wave inside each part of the scattering region whose size is comparable with the mean size of the irregularities present and that ordinary ray theory may be applied to calculate the travel times and amplitudes of both primary and scattered waves.

The theory supports the writer's earlier suggestion that observed precursors to the seismic core phase PKIKP may originate by scattering from the phases PKP_1 and PKP_2 by irregularities in the vicinity of the Earth's mantle-core boundary.

1. Introduction

The writer (Haddon, 1972) recently drew attention to certain inconsistencies between observational data on so-called precursors to the seismic core phase PKIKP and corresponding theoretical results entailed by previous interpretations of the precursors involving one or more transition layers surrounding the Earth's inner core. He suggested as an alternative interpretation that the precursors may originate by scattering from the main core phases PKP_1 and PKP_2 due to irregularities in the neighbourhood of the mantle-core boundary. Subsequently, Cleary and Haddon (1972) examined the new interpretation in some detail and assembled a body of evidence in support. Further support has recently been provided by Doornbos and Husebye (1972) from their analysis of precursor wavetrains recorded at the Norsar seismic array (see Haddon and Cleary, 1973). More recently, King (1973) has added further weight to the scattering interpretation with results from his analysis of precursor wavetrains recorded at the Warramunga seismic array.

In addition to accounting for precursors to PKIKP, seismic scattering may also account for several other observed

seismic "phases". For example, Cleary and Haddon (1973) have suggested that much of the body wave coda following P, including the so-called precursors to PP observed by Bolt and colleagues (1968) and others, may result from scattering in the crust and uppermost part of the upper mantle.

The main objection to the scattering interpretation raised so far has been that it has not been shown that the proposed scattering mechanism can account for the observed amplitudes of the phases in question. It is therefore of immediate importance to investigate the mechanism quantitatively. The present paper is a first step towards this end.

In this paper a simple theory of scattering of elastic body waves by small random fluctuations in density and elastic parameters is developed by appropriately generalising and extending certain aspects of the acoustic scattering theory given by Chernov (1960). The theory below not only establishes the plausibility of the proposed scattering mechanism, but also provides an adequate model for interpreting observational data quantitatively. For example, application of the theory has already shown that observed amplitudes of precursors to PKIKP can be accounted for by postulating random fluctuations in density and elastic parameters of order one per cent in the lowest 200 km of the mantle. Further numerical details relating to the application of the theory will be published in a separate paper.

2. Basic Equations for Scattering

Let u_i $(i = 1,2,3)$ denote the rectangular cartesian components of displacement of a point $P(x_i)$ in a perfectly elastic medium in which the density ρ, incompressibility k, and rigidity μ vary slightly from their mean values from point to point. The equations of motion for small displacements in such a medium may be written

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_i}\left((k - \frac{2}{3}\mu)\theta\right) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right], \tag{1}$$

where $\theta = \frac{\partial u_k}{\partial x_k}$ denotes the dilation.

The fluctuations in density and elastic parameters will be denoted by $\Delta\rho$, Δk and $\Delta\mu$, so that $\rho = \rho_0 + \Delta\rho$, $k = k_0 + \Delta k$ and $\mu = \mu_0 + \Delta\mu$, where ρ_0, k_0 and μ_0 denote the mean values and it is assumed that $|\Delta\rho| << \rho_0$, $|\Delta k| << k_0$ and $|\Delta\mu| << \mu_0$. For the present ρ_0, k_0 and μ_0 will be assumed to be constants. (In later sections, theory developed on this basis will be extended to cases where ρ_0, k_0 and μ_0 are slowly varying functions of distance from the Earth's centre.) Upon substituting for ρ, k and μ in equation (1), we obtain

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} - (k_0 + \frac{1}{3}\mu_0)\frac{\partial \theta}{\partial x_i} - \mu_0 \nabla^2 u_i$$

$$= -\Delta\rho \frac{\partial^2 u_i}{\partial t^2} + (\Delta k + \frac{1}{3}\Delta\mu)\frac{\partial \theta}{\partial x_i} + \Delta\mu\nabla^2 u_i \tag{2}$$

$$+ \frac{\partial}{\partial x_i}(\Delta k - \frac{2}{3}\Delta\mu)\theta + \frac{\partial}{\partial x_j}(\Delta\mu)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) .$$

When $\Delta\rho$, Δk and $\Delta\mu$ are all zero the equations (1) and (2) reduce to the usual equations of motion for a homogeneous medium. Let $u_i^{\,0}$ denote any solution of equation (2) when $\Delta\rho$, Δk and $\Delta\mu$ are all zero and let $u_i = u_i^{\,0} + u_i^{\,1}$ denote a corresponding perturbation solution of equation (2) when $\Delta\rho$, Δk and $\Delta\mu$ are non-zero. Upon substituting into equation (2), assuming that $|u_i^{\,1}| \ll |u_i^{\,0}|$, and ignoring terms of second and higher orders in small quantities, we obtain

$$\rho_0 \frac{\partial^2 u_i^{\,1}}{\partial t^2} - (k_0 + \frac{1}{3}\mu_0)\frac{\partial \theta^1}{\partial x_i} - \mu_0 \nabla^2 u_i^{\,1} = Q_i(\underset{\sim}{r}, t) , \tag{3}$$

where $\theta^1 = \dfrac{\partial u_k^{\,1}}{\partial x_k}$, $\theta^0 = \dfrac{\partial u_k^{\,0}}{\partial x_k}$, $\underset{\sim}{r} = (x_j)$

and

$$Q_i(\underset{\sim}{r},t) = -\Delta\rho \frac{\partial^2 u_i^{\,0}}{\partial t^2} + (\Delta k + \frac{1}{3}\Delta\mu)\frac{\partial\theta^0}{\partial x_i} + \Delta\mu\nabla^2 u_i^{\,0}$$

$$+ \frac{\partial}{\partial x_i}(\Delta k - \frac{2}{3}\Delta\mu)\theta^0 + \frac{\partial}{\partial x_j}(\Delta\mu)\left[\frac{\partial u_i^{\,0}}{\partial x_j} + \frac{\partial u_j^{\,0}}{\partial x_i}\right] . \tag{4}$$

Taking the divergence and curl of equation (3) gives

$$\frac{1}{\alpha_0^{\,2}}\frac{\partial^2\theta^1}{\partial t^2} - \nabla^2\theta^1 = \Theta(\underset{\sim}{r},t) , \tag{5}$$

and

$$\frac{1}{\beta_0^{\,2}}\frac{\partial^2\underset{\sim}{\xi}^1}{\partial t^2} - \nabla^2\underset{\sim}{\xi}^1 = \underset{\sim}{\Psi}(\underset{\sim}{r},t) , \tag{6}$$

where $\underset{\sim}{r} = (x_j)$, $\rho_0\alpha_0^{\,2} = k_0 + \frac{4}{3}\mu_0$, $\rho_0\beta_0^{\,2} = \mu_0$,

$$\underset{\sim}{\xi}^1 = \text{curl } (u_i^{\,1}) , \quad \Theta(\underset{\sim}{r},t) = \text{div } (Q_i)/(\rho_0\alpha_0^{\,2})$$

and $\underset{\sim}{\Psi}(\underset{\sim}{r},t) = \text{curl } (Q_i)/(\rho_0\beta_0^{\,2})$.

The resultants of the secondary scattered waves originating from within any *region* V of the medium are given by the following solutions of equations (5) and (6) (see, e.g., Stratton, pp. 424-428).

$$\theta^1 = \frac{1}{4\pi} \int_V \frac{1}{R'} \Theta(\underset{\sim}{r}', t') dV' , \tag{7}$$

$$\underset{\sim}{\xi}^1 = \frac{1}{4\pi} \int_V \frac{1}{R'} \underset{\sim}{\Psi}(\underset{\sim}{r}', t'') dV' , \tag{8}$$

where $\underset{\sim}{r}' = (x_i')$ denotes the source point, $\underset{\sim}{r} = (x_i)$ denotes the field point at which the solutions are to be evaluated at time t, $R' = |\underset{\sim}{r} - \underset{\sim}{r}'|$ is the distance between the source and field points and $t' = t - R'/\alpha_0$ and $t'' = t - R'/\beta_0$ denote retarded times.

The equations (5), (6), (7) and (8) show that under the influence of any primary wave u_i^0, each element of the inhomogeneous medium becomes effectively a source of scattered waves of both P and S types. In the following we shall restrict attention to scattered waves of P type originating from primary waves of P type.

3. P Wave Scattering from a Primary P Wave by Random Inhomogeneities

In general, the primary P wave disturbance may be represented by

$$u_i^{\,0} = \frac{\partial \Phi}{\partial x_i}\,, \qquad (i = 1,2,3)\,, \tag{9}$$

where $\Phi = \Phi(\underset{\sim}{r},t)$ satisfies the wave equation

$$\frac{1}{\alpha_0^{\,2}}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 0\,. \tag{10}$$

Upon substituting from equation (9) into equation (4) and using equation (10), we obtain

$$\begin{aligned} Q_i &= (\Delta k + \tfrac{4}{3}\,\Delta\mu - \alpha_0^{\,2}\Delta\rho)\,\Phi_{,ijj} \\ &\qquad + (\Delta k - \tfrac{2}{3}\,\Delta\mu)_{,i}\,\Phi_{,jj} + 2(\Delta\mu)_{,j}\,\Phi_{,ij}\,, \end{aligned} \tag{11}$$

where the subscripts following the commas denote partial differentiations

$$\text{e.g.} \qquad \Phi_{,ijj} \equiv \frac{\partial^3 \Phi}{\partial x_i\,\partial x_j^{\,2}}\,.$$

We shall now assume that the fluctuations $\Delta\rho$, Δk and $\Delta\mu$ are given by

$$\left.\begin{aligned} \Delta\rho/\rho_0 &= \ell H , \\ \Delta k/k_0 &= mH , \\ \text{and} \qquad \Delta\mu/\mu_0 &= nH , \end{aligned}\right\} \tag{12}$$

where ℓ, m and n are constants and $H = H(\underset{\sim}{r})$ denotes an isotropic stationary random function which has a correlation function N given by

$$N = N(r) = \langle H(\underset{\sim}{r}_1)H(\underset{\sim}{r}_2)\rangle , \tag{13}$$

where r here denotes $|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|$. In the present paper we shall further restrict consideration to the particular case where N is given by

$$N(r) = \exp(-r^2/\sigma^2) , \tag{14}$$

where σ is a positive constant called the correlation distance which characterises the scale of the random fluctuations in H. It may be noted that the corresponding root

mean square fluctuations in $\Delta\rho$, Δk and $\Delta\mu$ are given by

$$\left.\begin{aligned} \langle(\Delta\rho)^2\rangle^{\frac{1}{2}} &= \ell\rho_0 , \\ \langle(\Delta k)^2\rangle^{\frac{1}{2}} &= mk_0 , \\ \langle(\Delta\mu)^2\rangle^{\frac{1}{2}} &= n\mu_0 , \end{aligned}\right\} \tag{15}$$

so that ℓ, m and n represent the magnitudes of the root mean square proportional changes in ρ, k and μ.

Upon substituting from equation (12) into equation (11) and dividing through by $\rho_0\alpha_0^2$, we obtain

$$Q_i(\rho_0\alpha_0^2)^{-1} = (\gamma_1 - \gamma_2)H\Phi_{,ijj} + (\gamma_1 - \gamma_3)H_{,i}\Phi_{,jj} + \gamma_3 H_{,j}\Phi_{,ij} , \tag{16}$$

where

$$\left.\begin{aligned} \gamma_1 &= (mk_0 + \tfrac{4}{3}n\mu_0)/(\rho_0\alpha_0^2) = m + \tfrac{4}{3}(n - m)(\beta_0/\alpha_0)^2 , \\ \gamma_2 &= (\ell k_0 + \tfrac{4}{3}\ell\mu_0)/(\rho_0\alpha_0^2) = \ell , \\ \text{and} \quad \gamma_3 &= 2n\mu_0/(\rho_0\alpha_0^2) = 2n(\beta_0/\alpha_0)^2 . \end{aligned}\right\} \tag{17}$$

Taking the divergence of equation (16) gives

$$\Theta(\underset{\sim}{r},t) = a_1 H\Phi_{,iijj} + a_2 H_{,i}\Phi_{,ijj} + a_3 H_{,ij}\Phi_{,ij} + a_4 H_{,ii}\Phi_{,jj}\ , \tag{18}$$

where $a_1 = \gamma_1 - \gamma_2$, $a_2 = 2\gamma_1 - \gamma_2$,

$a_3 = \gamma_3$ and $a_4 = \gamma_1 - \gamma_3$.

Multiplying corresponding sides of equation (7) by their complex conjugates and averaging over the ensemble of possible distributions of H gives the following expression for the mean square amplitudes of scattered waves originating inside the region V.

$$<|\theta|^2> = \frac{1}{(4\pi)^2}\int_V\int_V \frac{1}{R'R''} <\Theta(\underset{\sim}{r}',t')\overline{\Theta(\underset{\sim}{r}'',t'')}> dV'dV''\ , \tag{19}$$

where $\underset{\sim}{r}' = (x_{i'})$ and $\underset{\sim}{r}'' = (x_{i''})$ denote the integration variables, $R' = |\underset{\sim}{r} - \underset{\sim}{r}'|$ and $R'' = |\underset{\sim}{r} - \underset{\sim}{r}''|$ denote the distances from the field to the source points, $t' = t - R'/\alpha_0$ and $t'' = t - R''/\alpha_0$ denote the retarded times and the overbar on $\Theta(r'',t'')$ denotes the complex conjugate.

4. *The Source Function for Scattered Waves*

Upon substituting from equation (18) into equation (19), we find that the right-hand side of the resulting equation can be expressed as a sum of integrals, a typical member of which is

$$I_{23} = \frac{1}{(4\pi)^2} \int\limits_V \int\limits_V \frac{a_2 a_3}{R'R''} N_{,i'i''j''} [\Phi_{,i'j'j'}][\overline{\Phi}_{,i''j''}] dV' dV'' \ , \tag{20}$$

where N is the correlation function given by equation (14) and the square brackets indicate that appropriate retarded times are to be taken.

Upon transforming the integration variables in each of the integrals like (20) by

$$\left.\begin{aligned} 2x_{j'} &= 2\overline{x}_j + \xi_j \ , \\ 2x_{j''} &= 2\overline{x}_j - \xi_j \ , \end{aligned}\right\} \quad j = 1,2,3 \tag{21}$$

we obtain, for example,

$$I_{23} = \frac{1}{(4\pi)^2} \iiint\limits_V d\overline{x}_1 d\overline{x}_2 d\overline{x}_3 \iiint\limits_{V_\xi} \frac{G_{23}(\overline{x}_\ell, \xi_\ell, x_i, t)}{R'R''} \frac{\partial^3 N}{\partial \xi_i^2 \partial \xi_j} d\xi_1 d\xi_2 d\xi_3 \ , \tag{22}$$

where, by equations (14) and (21), N is now given by

$$N = \exp\,[-\,(\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2)/\sigma^2]\ , \tag{23}$$

and where $G_{23}(\bar{x}_\ell, \xi_\ell, x_i, t)$ denotes the transformed form of $[\Phi_{,i'j'j'}][\bar{\Phi}_{,i''j''}]$ multiplied by $a_2 a_3$. In equation (22) the domain of integration V for $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is identical with the domains V for $(x_{1'}, x_{2'}, x_{3'})$ and $(x_{1''}, x_{2''}, x_{3''})$ while the domain V_ξ for (ξ_1, ξ_2, ξ_3) depends on $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. However, because of the presence of either the factor N or one of its derivatives in the integrands of each of the integrals like (20), the sum of which comprise the right-hand side of equation (19), each integrand will become small when $|\underset{\sim}{\xi}|$ becomes large compared with the correlation distance σ. It follows that if R', R" and the linear dimensions of V are all large compared with σ, then a satisfactory approximation to integrals like (20) can be obtained by replacing the factor R'R" by R^2, where $R^2 = \{(\bar{x}_1 - x_1)^2 + (\bar{x}_2 - x_2)^2 + (\bar{x}_3 - x_3)^2\}$, and the domain V_ξ by the infinite domain V_∞. We thus obtain, for example,

$$I_{23} = \frac{1}{(4\pi)^2} \iiint\limits_V \frac{d\bar{x}_1 d\bar{x}_2 d\bar{x}_3}{R^2} \iiint\limits_{V_\infty} \frac{\partial^3 N}{\partial \xi_i{}^2 \partial \xi_j} G(\bar{x}_\ell, \xi_\ell, x_i, t)\, d\xi_1 d\xi_2 d\xi_3\ ,$$

$$= \frac{1}{(4\pi)^2} \iiint\limits_V \frac{S_{23}}{R^2}\, d\bar{x}_1 d\bar{x}_2 d\bar{x}_3\ , \quad \text{say,} \tag{24}$$

where $$S_{23} = \iiint\limits_{V_\infty} \frac{\partial^3 N}{\partial \xi_i^{\,2} \partial \xi_j} G(\overline{x}_\ell, \xi_\ell, x_i, t) d\xi_1 d\xi_2 d\xi_3 .$$

Upon combining the complete set of terms constituting the right-hand side of equation (19), we obtain

$$\langle |\theta|^2 \rangle = \frac{1}{(4\pi)^2} \iiint\limits_{V} \frac{S}{R^2} d\overline{x}_1 d\overline{x}_2 d\overline{x}_3 , \tag{25}$$

where $$S = \sum_{m=1}^{4} \sum_{n=1}^{4} S_{mn} .$$

It is convenient to refer to S as the source function for scattered waves.

5. Approximation of the General Wave Function Φ by Plane Waves

For each particular point $\underset{\sim}{r} = \overline{\underset{\sim}{r}}^*$, say, the functions S_{mn} (above) depend (essentially) only on the values of the wave function Φ within the associated region of (ξ_1, ξ_2, ξ_3) where N and its derivatives are significantly large. We shall now assume that within each such limited region Φ can be adequately approximated by a plane wave function of the form

$$\Phi = \Phi\left(\underset{\sim}{n}\cdot(\underset{\sim}{r} - \overline{\underset{\sim}{r}}^*) + \alpha_0(\tau - t)\right) , \tag{26}$$

where $\underset{\sim}{r} = (x_i)$, $\overline{\underset{\sim}{r}}^* = (\overline{x}_i^*)$, $\underset{\sim}{n} = \underset{\sim}{n}(\overline{\underset{\sim}{r}}^*)$ denotes a unit vector in the direction of the normal to the wavefront passing through the point $\overline{\underset{\sim}{r}}^*$, and $\tau = \tau(\overline{\underset{\sim}{r}}^*)$ denotes the time taken for a wave to travel from its origin O to the point $\overline{\underset{\sim}{r}}^*$.

Now let $\underset{\sim}{x} = T\underset{\sim}{y}$ denote a transformation of coordinates from the x_i to a new "local" rectangular cartesian reference frame $O_1y_1y_2y_3$, which has its origin O_1 at the particular point $x_i = \overline{x}_i^*$ in x_i-space, its y_1 axis in the direction of the vector $\underset{\sim}{n}$ and which contains the field point P in the plane $y_3 = 0$. Since the equations (1) to

(19) are all independent of the particular cartesian reference frame used, these equations apply equally well when referred to a "new" x_i coordinate system whose axes coincide with those of the y_i system just introduced. In the new coordinate system equation (26) becomes

$$\Phi = \Phi\left(x_1 + \alpha_0(\tau - t)\right) , \tag{27}$$

where τ is the time taken for the primary wave to travel from its point of origin O to the origin $O_1(\bar{x}_i^{\,*})$ of the new coordinate system. For the remainder of this section the variables x_i, x_i', x_i'', etc. will all be taken to refer to the new coordinate system, unless stated otherwise.

Corresponding to equation (27) the equation (18) reduces to

$$\Theta(\underset{\sim}{r},t) = a_1 HF'' + a_2 \frac{\partial H}{\partial x_1} F' + a_3 \frac{\partial^2 H}{\partial x_1^{\,2}} F + a_4 \nabla^2 HF , \tag{28}$$

where $F = F\left(x_1 + \alpha_0(\tau - t)\right) \equiv \Phi''\left(x_1 + \alpha_0(\tau - t)\right)$ denotes the dilation θ_0 of the primary wave and the primes on F and Φ denote differentiations with respect to x_1. Upon substituting into equation (19) and using certain obvious symmetry properties, we obtain

$$<|\theta|^2> = \frac{1}{(4\pi)^2} \mathrm{Re} \int_V \int_V \frac{1}{R'R''} \Bigg\{$$

$$a_1{}^2 N F_1{}''\bar{F}_2{}'' + 2a_1a_2 \frac{\partial N}{\partial x_{1''}} F_1{}''\bar{F}_2{}' + 2a_1a_3 \frac{\partial^2 N}{\partial x_{1''}{}^2} F_1{}''\bar{F}_2$$

$$+ 2a_1a_4\nabla_2{}^2 N F_1{}''\bar{F}_2$$

$$+ a_2{}^2 \frac{\partial^2 N}{\partial x_{1'}\partial x_{1''}} F_1{}'\bar{F}_2{}' + 2a_2a_3 \frac{\partial^3 N}{\partial x_{1'}\partial x_{1''}{}^2} F_1{}'\bar{F}_2$$

$$+ 2a_2a_4 \frac{\partial}{\partial x_{1'}} \nabla_2{}^2 N F_1{}'\bar{F}_2$$

$$+ a_3{}^2 \frac{\partial^4 N}{\partial x_{1'}{}^2\partial x_{1''}{}^2} F_1\bar{F}_2 + 2a_3a_4 \frac{\partial^2}{\partial x_{1'}{}^2} \nabla_2{}^2 N F_1\bar{F}_2$$

$$+ a_4{}^2\nabla_1{}^2\nabla_2{}^2 N F_1\bar{F}_2\Bigg\} dV'dV'', \tag{29}$$

where $F_1 = F(s_1)$, $F_2 = F(s_2)$, $F_1{}' = dF(s_1)/ds_1$, etc.,

$$s_1 = x_{1'} + R' + \alpha_0(\tau - t), \qquad s_2 = x_{1''} + R'' + \alpha_0(\tau - t),$$

and R' and R'' *again* denote the distances between the source and field points.

Upon transforming the variables $(x_{i'})$ and $(x_{i''})$ in equation (29) by the transformation (21) and approximating $R'R''$ by R^2 *and* V_ξ by V_∞, as before, we obtain

$$<|\theta|^2> \;=\; \frac{1}{(4\pi)^2}\iiint\limits_{V} \frac{S}{R^2}\, d\bar{x}_1 d\bar{x}_2 d\bar{x}_3 \;, \tag{30}$$

where

$$\begin{aligned} S = \mathrm{Re} \iiint\limits_{V_\infty} \Big\{ & a_1^{\,2} N F_1'' \bar{F}_2'' - 2a_1 a_2 \frac{\partial N}{\partial \xi_1} F_1'' \bar{F}_2' + 2a_1 a_3 \frac{\partial^2 N}{\partial \xi_1^{\,2}} F_1'' \bar{F}_2 \\ & \qquad + 2a_1 a_4 \nabla^2 N F_1'' \bar{F}_2 \\ & - a_2^{\,2} \frac{\partial^2 N}{\partial \xi_1^{\,2}} F_1' \bar{F}_2' + 2a_1 a_3 \frac{\partial^3 N}{\partial \xi_1^{\,3}} F_1' \bar{F}_2 + 2a_2 a_4 \frac{\partial}{\partial \xi_1} \nabla^2 N F_1' \bar{F}_2 \\ & + a_3^{\,2} \frac{\partial^4 N}{\partial \xi_1^{\,4}} F_1 \bar{F}_2 + 2a_3 a_4 \frac{\partial^2}{\partial \xi_1^{\,2}} \nabla^2 N F_1 \bar{F}_2 \\ & + a_4^{\,2} \nabla^2 \nabla^2 N F_1 \bar{F}_2 \Big\} d\xi_1 d\xi_2 d\xi_3 \;, \end{aligned} \tag{31}$$

where ∇^2 denotes $\frac{\partial^2}{\partial \xi_1^{\,2}} + \frac{\partial^2}{\partial \xi_2^{\,2}} + \frac{\partial^2}{\partial \xi_3^{\,2}}$ and s_1 and s_2 are now given by

$$\left.\begin{aligned} s_1 &= \bar{x}_1 + \tfrac{1}{2}\xi_1 + R' + \alpha_0(\tau - t) \;, \\ \text{and} \qquad s_2 &= \bar{x}_1 - \tfrac{1}{2}\xi_1 + R'' + \alpha_0(\tau - t) \;. \end{aligned}\right\} \tag{32}$$

Although the equations (25) and (30) are identical in form, their derivations show that they have distinctly different meanings. For, apart from referring to different coordinate systems, the function Φ given by equation (26) has been assumed to approximate the general function Φ involved in equation (25) only in the vicinity of the particular point $x_i = \bar{x}_i^*$ in the original coordinate system. In respect of the functions S given by equations (25) and (31), however, it is evident that under the assumed circumstances the value of the function S given by equation (31) at the point $\bar{x}_i = 0$ in the new coordinate system will approximate the value of the function S in equation (25) at the point $\bar{x}_i = \bar{x}_i^*$ in the original coordinate system to the extent that the function Φ given by equation (26) approximates the general function Φ involved in equation (25) inside the region where N and its derivatives are significant. The equation (31) for S evaluated at the point $\bar{x}_i = 0$ therefore provides an approximation to the function S in (25) at the point $\bar{x}_i^*$ in the original coordinate system. In the next section we shall reduce the expression (31) for S to a simpler analytical form.

6. Transformation of the Source Function Integral

Let $\underset{\sim}{r}'$ denote the position vector O_1P' of the source point $P'(x_{i'})$ referred to the local coordinate system introduced in the previous section, R the distance O_1P from the origin O_1 to the field point $P(x_i)$, $\underset{\sim}{n}_1$ a unit vector in the direction O_1P and ϕ the angle between $\underset{\sim}{n}$ and $\underset{\sim}{n}_1$, as shown in figure 1.

Since $|\underset{\sim}{r}'| << R$ and O_1P is in the plane $x_3 = 0$ (by choice of coordinate system) we find that to sufficient accuracy

$$R' = R - \underset{\sim}{n}_1 \cdot \underset{\sim}{r}' = R - x_{1'} \cos\phi - x_{2'} \sin\phi ,$$

and similarly

$$R'' = R - \underset{\sim}{n}_1 \cdot \underset{\sim}{r}'' = R - x_{1''} \cos\phi - x_{2''} \sin\phi . \qquad (33)$$

Transforming these expressions by use of the equations (21), then substituting into equation (32) and putting $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$ we find that

$$s_1 = p\xi_1 + q\xi_2 + \alpha_0\tau' ,$$

and

$$s_2 = -p\xi_1 - q\xi_2 + \alpha_0\tau' , \qquad (34)$$

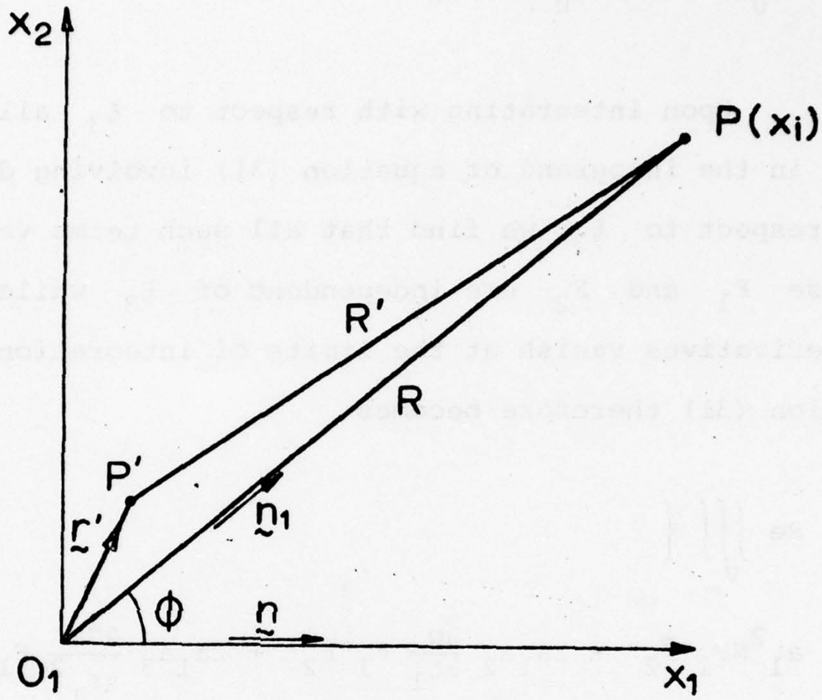

Fig. 1 The local coordinate system for evaluating the source function S.

where $\quad p = \sin^2\phi/2$, $\quad q = -\sin\phi/2\cos\phi/2$

and $\quad \alpha_0\tau' = R + \alpha_0(\tau - t)$.

Upon integrating with respect to ξ_3 all those terms in the integrand of equation (31) involving derivatives with respect to ξ_3, we find that all such terms vanish because F_1 and F_2 are independent of ξ_3 while N and its derivatives vanish at the limits of integration. The equation (31) therefore becomes

$$
\begin{aligned}
S = \mathrm{Re} \iiint\limits_{V_\infty} \Bigg\{ & \\
& a_1{}^2 N F_1{}''\bar{F}_2{}'' - 2a_1a_2 \frac{\partial N}{\partial \xi_1} F_1{}''\bar{F}_2{}' + 2a_1a_3 \frac{\partial^2 N}{\partial \xi_1{}^2} F_1{}''\bar{F}_2 \\
& \qquad + 2a_1a_4 \left(\frac{\partial^2}{\partial \xi_1{}^2} + \frac{\partial^2}{\partial \xi_2{}^2}\right) N F_1{}''\bar{F}_2 \\
& - a_2{}^2 \frac{\partial^2 N}{\partial \xi_1{}^2} F_1{}'\bar{F}_2{}' + 2a_2a_3 \frac{\partial^3 N}{\partial \xi_1{}^3} F_1{}'\bar{F}_2 \\
& \qquad + 2a_1a_4 \frac{\partial}{\partial \xi_1}\left(\frac{\partial^2}{\partial \xi_1{}^2} + \frac{\partial^2}{\partial \xi_2{}^2}\right) N F_1{}'\bar{F}_2 \\
& + a_3 \frac{\partial^4 N}{\partial \xi_1{}^4} F_1\bar{F}_2 + 2a_3a_4 \frac{\partial^2}{\partial \xi_1{}^2}\left(\frac{\partial^2}{\partial \xi_1{}^2} + \frac{\partial^2}{\partial \xi_2{}^2}\right) N F_1\bar{F}_2 \\
& + a_4{}^2 \left(\frac{\partial^4}{\partial \xi_1{}^4} + \frac{2\partial^4}{\partial \xi_1{}^2\partial \xi_2{}^2} + \frac{\partial^4}{\partial \xi_2{}^4}\right) N F_1\bar{F}_2 \Bigg\} d\xi_1 d\xi_2 d\xi_3 \ ,
\end{aligned}
\tag{35}
$$

where $F_1 = F(s_1)$, etc., and s_1 and s_2 are given by equations (34).

Integrating the various terms in equation (35) by parts with respect to ξ_1 and ξ_2, as appropriate, and using the results that

$$q \frac{\partial}{\partial \xi_1}(F_1 \bar{F}_2) = p \frac{\partial}{\partial \xi_2}(F_1 \bar{F}_2), \qquad q^2 \frac{\partial^2}{\partial \xi_1{}^2}(F_1 \bar{F}_2) = p^2 \frac{\partial^2}{\partial \xi_2{}^2}(F_1 \bar{F}_2),$$

etc., we obtain

$$\begin{aligned} S \;=\; \mathrm{Re} \iiint\limits_{V_\infty} N\Big\{ & a_1{}^2 F_1'' \bar{F}_2'' + 2a_1a_2 \frac{\partial}{\partial \xi_1}(F_1'' \bar{F}_2') + 2a_1a_5 \frac{\partial^2}{\partial \xi_1{}^2}(F_1'' \bar{F}_2) \\ & - a_2{}^2 \frac{\partial^2}{\partial \xi_1{}^2}(F_1' \bar{F}_2') - 2a_2a_5 \frac{\partial^3}{\partial \xi^3}(F_1' \bar{F}_2) \\ & + a_5{}^2 \frac{\partial^4}{\partial \xi_1{}^4}(F_1 \bar{F}_2)\; d\xi_1 d\xi_2 d\xi_3 \;, \end{aligned} \tag{36}$$

where $a_5 = (a_3 + a_4) + a_4 q^2/p^2$ and the other symbols are as previously defined.

After some further manipulations using results like

$$\mathrm{Re} \iiint\limits_{V_\infty} N \frac{\partial}{\partial \xi_1}(F_1' \bar{F}_2'')\, d\xi_1 d\xi_2 d\xi_3 = -\, \mathrm{Re} \iiint\limits_{V_\infty} N \frac{\partial}{\partial \xi_1}(F_1'' \bar{F}_2')\, d\xi_1 d\xi_2 d\xi_3 \,,$$

we finally obtain

$$S \;=\; \mathrm{Re} \iiint\limits_{V_\infty} N\left\{c_1F_1''\overline{F}_2'' + c_2F_1'''\,\overline{F}_2' + c_3F_1''''\overline{F}_2\right\}d\xi_1 d\xi_2 d\xi_3 \;, \qquad (37)$$

where

$$\left.\begin{aligned}
c_1 &= a_1^{\,2} - 2pa_2(a_1 - pa_2) + 2p^2a_5(a_1 - 3(pa_2 - p^2a_5))\;, \\
c_2 &= 2pa_2(a_1 - pa_2) + 4p^2a_5(-\,a_1 + 2(pa_2 - p^2a_5))\;, \\
\text{and}\quad c_3 &= 2p^2a_5(a_1 - (pa_2 - p^2a_5))\;.
\end{aligned}\right\} \qquad (38)$$

7. Summary

When the general primary wave disturbance Φ (equation (9)) can be adequately approximated by plane waves of the form of equation (26) within each subregion of V whose linear dimensions are of order σ, then the mean square amplitude of scattered waves originating inside V and arriving at the field point P at time t is given by equation (25) where the source function S is given by equation (37).

8. Scattering from a Simple Harmonic Primary Wave

In this section we obtain an explicit expression for the source function S (equation (37)) for the case when at each fixed point $\underset{\sim}{r} = \underset{\sim}{\bar{r}}$ of the scattering region V the primary wave can be represented by

$$\theta_0 = A_0 \exp(iks) , \tag{39}$$

where $s = \underset{\sim}{n} \cdot (\underset{\sim}{r} - \underset{\sim}{\bar{r}}) + \alpha_0(\tau - t)$ and where the unit normal vector $\underset{\sim}{n}$, the wavenumber k and the amplitude A_0 are assumed to vary relatively slowly with $\underset{\sim}{\bar{r}}$, and $\tau = \tau(\underset{\sim}{\bar{r}})$ denotes the time taken for the wave to travel from its source to the point $P(\underset{\sim}{\bar{r}})$, as before. For this case we have

$$\left.\begin{aligned} F(s_1) &= A_0 \exp(iks_1) , \\ F(s_2) &= A_0 \exp(iks_2) , \end{aligned}\right\} \tag{40}$$

where s_1 and s_2 are given by the equations (34).

Upon substituting from equations (23) and (40) into equation (37), we obtain

$$S = (c_1 - c_2 + c_3)k^4|A_0|^2 \iiint_{V_\infty} \exp\Big(-\left(\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2\right)/\sigma^2 + 2ik\left(p\xi_1 + q\xi_2\right)\Big) d\xi_1 d\xi_2 d\xi_3 ,$$

$$= (c_1 - c_2 + c_3)k^4|A_0|^2\sigma^3(\sqrt{\pi})^3 \exp(-k^2\sigma^2\sin^2\phi/2). \qquad (40)$$

From the equations (18) and (38) we find that

$$(c_1 - c_2 + c_3) = (a_1 - 2pa_2 + 4p^2a_5)^2 ,$$

$$= (\gamma_1 - \gamma_2\cos\phi - \gamma_3\sin^2\phi)^2 ,$$

$$= \Gamma^2 , \text{ say ,} \qquad (41)$$

where γ_1, γ_2 and γ_3 are given by the equations (17).

Substituting for S from equation (40) into equation (25), we obtain

$$<|\theta|^2> = \frac{1}{16\sqrt{\pi}} \int_V \frac{\Gamma^2|A_0|^2k^4\sigma^3}{R^2} \exp(-k^2\sigma^2\sin^2\phi/2)dV , \qquad (42)$$

where dV denotes $d\bar{x}_1 d\bar{x}_2 d\bar{x}_3$.

When $|A_0|$, R, ϕ, etc., vary significantly throughout the region V, the equation (42) would generally need to be evaluated numerically. If, however, equation (39) adequately represents the primary wave throughout the whole of V with k, A_0, and $\underset{\sim}{n}$ all practically constant, and if also R is large compared with the linear dimensions of V, then the integrand of (42) will be practically constant and we immediately obtain

$$<|\theta|^2> = \frac{V\Gamma^2|A_0|^2k^4\sigma^3}{16\sqrt{\pi}R^2}\exp(-k^2\sigma^2\sin^2\phi/2) , \qquad (43)$$

where V here denotes the volume of the scattering region.

For the particular case of a fluid medium the equation (43) reduces to agreement with a corresponding result given by Chernov (p. 52).

9. Scattering from a Random Primary Wave

The observed wavetrain following P frequently appears to have a random phase character. In the present section we therefore consider the scattering from a primary wave which at each fixed point $\underset{\sim}{r} = \underset{\sim}{\bar{r}}$ of the scattering region V can be represented by

$$\theta_0 = A_0 F(s) \ , \tag{44}$$

where $s = \underset{\sim}{n} \cdot (\underset{\sim}{r} - \underset{\sim}{\bar{r}}) + \alpha_0(\tau - t)$, as before, A_0 denotes the root mean square amplitude, and where F here denotes a real stationary random function with a correlation function M given by

$$M = \langle F(s_1)F(s_2)\rangle = \exp\left(-(s_1 - s_2)^2/\lambda^2\right) . \tag{45}$$

In this equation λ denotes the correlation parameter characterising the range and distribution of wavelengths in the signal.

Taking the statistical average of equation (37) over the ensemble of random functions F and using (45), we obtain

$$\langle S\rangle = A_0^{\,2} \iiint\limits_{V_\infty} N\left\{c_1 \frac{\partial^4 M}{\partial s_1^{\,2}\partial s_2^{\,2}} + c_2 \frac{\partial^4 M}{\partial s_1^{\,3}\partial s_2} + c_3 \frac{\partial^4 M}{\partial s_1^{\,4}}\right\} d\xi_1 d\xi_2 d\xi_3 \ ,$$

$$= (c_1 - c_2 + c_3) A_0^{\,2} \iiint N \frac{\partial^4 M}{\partial s^4} d\xi_1 d\xi_2 d\xi_3 \ , \qquad (46)$$

where s_1 and s_2 are given by the equations (34) and $s = s_1 - s_2$.

Substituting for M and N from equations (23) and (45) and using (34), we obtain

$$\langle S\rangle = \frac{4\Gamma^2 A_0^{\,2}}{\lambda^4} \iiint\limits_{V_\infty} (3 - 12s^2/\lambda^2 + 4s^4/\lambda^4)$$
$$\times \exp\left[-(\xi_1^{\,2} + \xi_2^{\,2} + \xi_3^{\,2})/\sigma^2 - s^2/\lambda^2\right] d\xi_1 d\xi_2 d\xi_3 \ , \qquad (47)$$

where $s = 2p\xi_1 + 2q\xi_2$.

Changing variables in equation (47) by

$$\left.\begin{aligned} u &= \xi_1 \sin\phi/2 - \xi_2 \cos\phi/2 \ , \\ v &= \xi_1 \cos\phi/2 + \xi_2 \sin\phi/2 \ , \\ w &= \xi_3 \ , \end{aligned}\right\} \qquad (48)$$

we obtain

$$\langle S\rangle = \frac{4\Gamma^2 A_0{}^2}{\lambda^4} \iiint\limits_{V_\infty} (3 - 48pu^2/\lambda^2 + 64p^2u^4/\lambda^4)$$

$$\times \exp\left(- u^2\left(\frac{1}{\sigma^2} + \frac{4p}{\lambda^2}\right) - (v^2 + w^2)/\sigma^2\right) dudvdw \; . \tag{49}$$

Integrating (49) gives

$$\langle S\rangle = \frac{12\Gamma^2 A_0{}^2 (\sqrt{\pi})^3 \sigma^3}{\lambda^4\left(1 + \frac{4\sigma^2}{\lambda^2}\sin^2\phi/2\right)^{5/2}} , \tag{50}$$

and substituting into equation (25) gives

$$\langle\theta^2\rangle = \frac{3}{4\sqrt{\pi}} \int\limits_V \frac{\Gamma^2 A_0{}^2 \sigma^3}{R^2\lambda^4\left(1 + \frac{4\sigma^2}{\lambda^2}\sin^2\phi/2\right)^{5/2}} \, dV \; . \tag{51}$$

As in the previous section, if the integrand of this expression varies significantly throughout the region V, then the expression would generally need to be evaluated numerically. If the integrand remains practically constant throughout V then we obtain

$$\langle\theta^2\rangle = \frac{3V\Gamma^2 A_0{}^2 \sigma^3}{4\sqrt{\pi}R^2\lambda^4\left(1 + \frac{4\sigma^2}{\lambda^2}\sin^2\phi/2\right)^{5/2}} , \tag{52}$$

where V again denotes the volume of the scattering region.

10. Scattering from a Wavepacket of Finite Length

We here consider scattering from a primary wave disturbance of the form

$$\theta_0 = F(s) = A_0 E(s) G(s) \, , \tag{53}$$

where $s = \underset{\sim}{n} \cdot (\underset{\sim}{r} - \overline{\underset{\sim}{r}}) + \alpha_0(\tau - t)$, $E(s)$ represents the envelope shape of the wavepacket and $A_0 G(s)$ represents either the simple harmonic wavetrain given by equation (39) or the random wavetrain given by equation (44). Differentiating equation (53) gives

$$F'(s) = \left\{1 + \frac{E'(s)}{E(s)} \Big/ \left(\frac{G'(s)}{G(s)}\right)\right\} A_0 E(s) G'(s) \, . \tag{54}$$

If $E(s)$ is a relatively slowly varying function compared with $G(s)$ we may approximate in equations such as (54) by neglecting the derivatives of $E(s)$. For example, if $E(s) = e^{-s^2/L^2}$ and $G(s) = e^{iks}$ then the second term in the brackets in equation (54) will generally be less than $\frac{2}{Lk}$, which is relatively small compared with unity whenever the wavepacket contains several cycles.

Under the assumed circumstances we readily obtain

$$<|\theta|^2> \;=\; \frac{1}{16\sqrt{\pi}} \int_V \frac{\Gamma^2 |A_0 E|^2 k^4 \sigma^3}{R^2} \exp(-\,k^2\sigma^2 \sin^2\phi/2)\,dV\ , \tag{55}$$

and

$$<\theta^2> \;=\; \frac{3}{4\sqrt{\pi}} \int_V \frac{\Gamma^2 |A_0 E|^2 \sigma^3}{R^2 \lambda^4 \left(1 + \frac{4\sigma^2}{\lambda^2} \sin^2\phi/2\right)^{5/2}}\,dV\ , \tag{56}$$

corresponding to the equations (42) and (51), respectively, where $E = E(\alpha_0\tau')$ and $\alpha_0\tau' = R + \alpha_0(\tau - t)$. It may be noted that the integrands of equations (55) and (56) are significant only in those parts of V where $E(\alpha_0\tau')$ is significant. The scattering region may therefore be taken infinite when appropriate.

11. Scattering from a Pulse-Like Primary Wave

As a final example of the application of equation (37), we consider the scattering from a primary wave pulse which at each fixed point $\underset{\sim}{r} = \overline{\underset{\sim}{r}}$ of the scattering region can be represented by

$$\theta_0 = A_0 \exp(-s^2/\lambda^2) , \tag{57}$$

where $s = \underset{\sim}{n} \cdot (\underset{\sim}{r} - \overline{\underset{\sim}{r}}) + \alpha_0(\tau - t)$ and λ here characterises the length of the pulse. In this case we must evaluate equation (37) with

$$\left.\begin{aligned} F_1 &= F(s_1) = A_0 \exp(-s_1{}^2/\lambda^2) , \\ \text{and} \quad F_2 &= F(s_2) = A_0 \exp(-s_2{}^2/\lambda^2) , \end{aligned}\right\} \tag{58}$$

where s_1 and s_2 are given by the equations (34).

Consider, for example, the integration of the first term in the integrand of the expression (37).

$$I_1 = \iiint_{V_\infty} NF_1''F_2''d\xi_1 d\xi_2 d\xi_3 \ ,$$

$$= \frac{A_0^{\ 2}}{\lambda^4} \iiint_{V_\infty} \left(- 2 + \frac{4s_1^{\ 2}}{\lambda^2}\right)\left(- 2 + \frac{4s_2^{\ 2}}{\lambda^2}\right) \exp\left(- (\xi_1^{\ 2} + \xi_2^{\ 2} + \xi_3^{\ 2})/\sigma^2 - (s_1^{\ 2} + s_2^{\ 2})/\lambda^2\right) d\xi_1 d\xi_2 d\xi_3 \ . \tag{59}$$

Changing variables to u, v, w by using the equations (48) gives

$$I_1 = \frac{A_0^{\ 2}}{\lambda^4} \exp(- 2X^2) \iiint_{V_\infty} \left(4 - 16\left(\frac{pu^2}{\lambda^2} + X^2\right) + 16\left(\frac{p^2u^4}{\lambda^4} - \frac{2pX^2u^2}{\lambda^2} + X^4\right)\right) \times \exp\left(- \left(\frac{1}{\sigma^2} + \frac{2p}{\lambda^2}\right)u^2 - (v^2 + w^2)/\sigma^2\right) dudvdw \ , \tag{60}$$

where $X = \frac{\alpha_0\tau'}{\lambda}$, $p = \sin^2\phi/2$ and $\alpha_0\tau' = R + \alpha_0(\tau - t)$.

Now integrating with respect to u, v and w gives

$$I_1 = \frac{4(\sqrt{\pi})^3\sigma^3 A_0^{\ 2} \exp(- 2X^2)}{\lambda^4 Y^{5/2}} \left(Y^2(1 - 2X^2)^2 + Y(1 - Y)(1 + 2X^2) + 3p^2\sigma^4/\lambda^4\right) \tag{61}$$

where $Y = 1 + \frac{2\sigma^2}{\lambda^2} \sin^2\phi/2.$

Upon similarly integrating the other terms in equation (37) and combining, we obtain

$$S = \frac{4(\sqrt{\pi})^3 \sigma^3 A_0{}^2}{\lambda^4 Y^{5/2}} \exp(-2X^2)(g_1 X^4 + g_2 X^2 + g_3) , \tag{62}$$

$$\left.\begin{aligned} \text{where} \quad g_1 &= 4a_1{}^2 Y^2 , \\ g_2 &= (-6a_1{}^2 Y^2 + 2Y(c_1 - 3c_3)) , \\ g_3 &= \frac{3}{4}(1 - Y)^2 \Gamma^2 + \frac{3}{2} c_2 Y^2 + (c_1 - \frac{3}{2} c_2 + 3c_3) Y , \end{aligned}\right\} \tag{63}$$

and X and Y have the same meanings as in equations (60) and (61).

By equations (25) and (62) the resultant mean square amplitude of scattered waves arriving at P from V is given by

$$\langle \theta^2 \rangle = \frac{1}{4\sqrt{\pi}} \int_V \frac{\sigma^3 A_0{}^2 (g_1 X^4 + g_2 X^2 + g_3)}{R^2 \lambda^4 Y^{5/2}} \exp(-2X^2) dV . \tag{64}$$

12. Application of Scattering Theory to Spherically Symmetrical Earth Models

Consider an Earth model which is spherically symmetrical except for small random fluctuations in the density and elastic parameters inside a certain region V. Let ρ_0, k_0, μ_0, α_0 and β_0 denote the mean density, incompressibility, rigidity and P and S velocities at distance r from the centre of the model. In the present section we shall derive a formula for the resulting P type scattering when a primary P wave propagates through the irregular region V. We shall assume that the primary wave originates from a spherically symmetrical point source A and that ordinary ray theory (see, e.g., Bullen (1963), chapters 7 and 8) may be applied to calculate travel times and amplitudes of both the primary wave and the secondary scattered waves excited by the primary wave.

Let I denote the energy in the primary wave emitted per unit solid angle from the source A and let e_A be the angle which any ray makes with the level surface $r = r_A$ through A (see figure 2). Then the energy E per unit area of the portion of the wavefront emerging at an angle e_B to the level surface $r = r_B$ through any point B is given by (Bullen, p. 126)

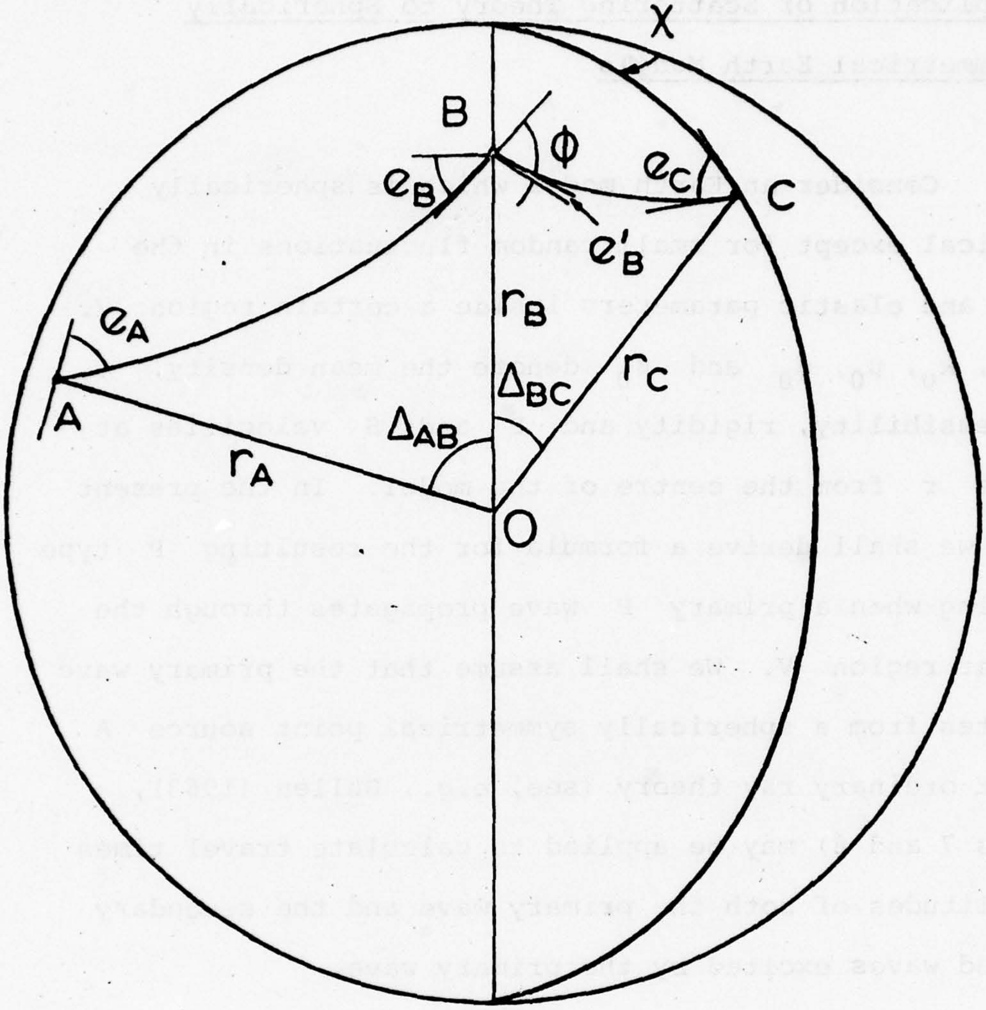

Fig. 2 The primary wave ray AB and the scattered wave ray BC for a primary source A, a scattering point B and a receiver C in a spherically symmetrical earth model: χ is the angle of intersection of the diametral planes containing AB and BC; e_A, e_B, e_B' and e_C are the angles at which the rays intersect the level surfaces through A, B and C.

$$E = \frac{I}{r_B^{\ 2}} \frac{\cos e_A}{\sin e_B \sin \Delta_{AB}} \left| \frac{de_A}{d\Delta_{AB}} \right| , \tag{65}$$

where Δ_{AB} denotes the angle AOB subtended by A and B at the centre O of the model and the range e_A to $e_A + de_A$ corresponds to the range Δ_{AB} to $\Delta_{AB} + d\Delta_{AB}$.

For a simple harmonic wavetrain the amplitude A of the wave is related to the energy E per unit area of wavefront by (Bullen, p. 128)

$$A^2 = f.E , \tag{66}$$

where the factor f depends on the wavelength and period of the wave and the density of the medium. A similar relationship holds generally for plane waves of similar waveforms.

From equations (65) and (66) we obtain

$$A_B^{\ 2} = f_B \frac{I}{r_B^{\ 2}} \frac{\cos e_A}{\sin e_B \sin \Delta_{AB}} \left| \frac{de_A}{d\Delta_{AB}} \right| , \tag{67}$$

where A_B is the amplitude of the primary wave at B and f_B is the value of the factor f at B. The expression (67) will be applied shortly to give the amplitude of the primary wave in scattering formulas such as (55), (56) and (64). We shall first consider effects of the model's velocity structure on amplitudes of scattered waves.

The equations (55) (56) and (64) can all be written in the form

$$<|\theta|^2> = \frac{1}{(4\pi)^2} \sum_V \frac{S\Delta V}{R^2}, \tag{68}$$

where ΔV represents an infinitesimal volume element of V. The resultant of the scattered waves arriving at the field point P may thus be regarded as the sum of contributions originating in the volume elements ΔV comprising V. The contribution from a single element ΔV may be represented by

$$<\Delta\theta^2> = \frac{S\Delta V}{(4\pi R)^2}. \tag{69}$$

The energy per unit area of wavefront arriving at the field point P from ΔV is given by

$$\Delta E = \frac{S\Delta V}{f_B(4\pi R)^2}. \tag{70}$$

This contribution may be regarded as coming from a point source B within ΔV which emits $S\Delta V/(f_B \times (4\pi)^2)$ units of energy per unit solid angle in the direction BP. The equations (55), (56) and (57) all apply only to the case where ρ_0, k_0 and μ_0 are all constants throughout the

region connecting B and P. We shall now extend these results to the case of a spherically symmetrical model where ρ_0, k_0 and μ_0 vary with r.

If a point source of intensity $S\Delta V/\left(f_B \times (4\pi)^2\right)$ were located at a point B in a spherically symmetrical model then the energy ΔE_C per unit area of wavefront originating at B and passing through a field point C would be given by (cf. Bullen, p. 126)

$$\Delta E_C = \frac{S\Delta V}{(4\pi)^2 f_B r_C^{\,2} \eta_B} \frac{\cot e_B{}'}{\sin e_C \sin \Delta_{BC}} \left| \frac{d^2 T_{BC}}{d\Delta^2_{BC}} \right| , \tag{71}$$

where $e_B{}'$ is the angle which any ray leaving B makes with the level surface $r = r_B$ through B, e_C is the angle which this ray makes with the level surface $r = r_C$ through C, T_{BC} is the travel time of a wave travelling from B to C along the ray, Δ_{BC} is the angle BOC subtended by B and C at the centre O of the model and $\eta_B = r_B/\alpha_B$. Thus in order to allow for geometrical focusing effects associated with velocity structure in the Earth model we must replace the equation (70) by the equation (71). The corresponding contribution to the mean square amplitude of the waves arriving at C is then given by

$$\langle \Delta\theta^2 \rangle = \frac{f_C S \Delta V \cot e_B{}'}{(4\pi)^2 f_B r_C{}^2 \eta_B \sin e_C \sin \Delta_{BC}} \left| \frac{d^2 T_{BC}}{d\Delta^2{}_{BC}} \right| , \qquad (72)$$

where f_C is the value of the factor f at C. The resultant mean square amplitude at C from all the elementary sources ΔV then becomes

$$\langle \theta^2 \rangle = \frac{1}{(4\pi)^2} \int_V \frac{f_C \cot e_B{}'}{f_B r_C{}^2 \eta_B \sin e_C \sin \Delta_{BC}} \left| \frac{d^2 T_{BC}}{d\Delta^2{}_{BC}} \right| S dV , \qquad (73)$$

where S denotes the source function appropriate to the particular primary wave assumed (e.g. equations (40), (50) or (62)).

The equation (73) will now be referred to a coordinate system which is particularly suitable for numerical evaluation of the integral. Let ψ_B denote the angle between any fixed diametral reference plane passing through A, and the diametral plane passing through the focal point A and the scattering point B. We shall take as independent variables ψ_B, e_A and r_B. (The particular advantage of this coordinate system is that the surfaces defined by $\psi_B = \psi_1$, $\psi_B = \psi_2$, $e_A = e_1$ and $e_A = e_2$ define a tube of rays emerging from the focus A.) In the coordinate system taken the volume element dV_B bounded by the surfaces ψ_B, $\psi_B + d\psi_B$, e_A, $e_A + de_A$, r_B and $r_B + dr_B$ is given by

$$dV_B = r_B^2 \sin \Delta_{AB} d\Delta_{AB} d\psi_B dr_B \,, \tag{74}$$

where $d\Delta_{AB}$ corresponds to de_A. Combining equations (67) and (74) gives

$$A_B^2 dV_B = f_B I \frac{\cos e_A}{\sin e_B} de_A d\psi_B dr_B \,. \tag{75}$$

Finally, writing $S = S_B A_B^2$ and substituting into equation (73) gives

$$\langle\theta^2\rangle = \frac{f_C I}{(4\pi)^2 r_C^2} \iiint\limits_{V_1} \frac{\cos e_A \cot e_B'}{\eta_B \sin e_B \sin e_C \sin \Delta_{BC}} \left|\frac{d^2 T_{BC}}{d\Delta^2_{BC}}\right| S_B de_A d\psi_B dr_B \,, \tag{76}$$

where the domain V_1 for (e_A, ψ_B, r_B) corresponds with the domain V for $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$.

13. Scattering from Wavepackets in a Spherically Symmetrical Model

For scattering from a wavepacket of the form of equation (53), where $A_0G(s)$ is given by either equation (39) or (44), the equation (76) gives

$$<|\theta|^2> \;=\; \frac{f_C I}{16\sqrt{\pi}r_C{}^2}\iiint_{V_1}\frac{\Gamma^2[E(\alpha_0\tau')]^2k^4\sigma^3\ \cos e_A\ \cot e_B{}'}{\eta_B\ \sin e_B\ \sin e_C\ \sin\Delta_{BC}} \times \left|\frac{d^2T_{BC}}{d\Delta^2{}_{BC}}\right|^{\leq 0} \exp(-\ k^2\sigma^2\sin^2\phi/2)\,de_A d\psi_B dr_B\ , \tag{77}$$

and

$$<0^2> \;=\; \frac{3f_C I}{4\sqrt{\pi}r_C{}^2}\iiint_{V_1}\frac{\Gamma^2[E(\alpha_0\tau')]^2\sigma^3}{\eta_B\ \lambda^4\left(1+\dfrac{4\sigma^2}{\lambda^2}\sin^2\phi/2\right)^{5/2}} \times \frac{\cos e_A\ \cot e_B{}'}{\sin e_B\ \sin e_C\ \sin\Delta_{BC}}\left|\frac{d^2T_{BC}}{d\Delta^2{}_{BC}}\right| de_A d\psi_B dr_B\ , \tag{78}$$

corresponding to equations (55) and (56) respectively, where τ' is now given by

$$\tau' \;=\; T_{AB} + T_{BC} - t\ , \tag{79}$$

and α_0 denotes the P velocity at the point (e_A, ψ_B, r_B).

While equations (77) and (78) are in an ideal form for numerical integration in some applications, in others where scattering may occur at or near the lowest points of the primary wave rays, it is more suitable to change from the variable r_B to s_B, where s_B denotes the length of the ray path from A to B. This transformation may be accomplished simply by substituting ds_B for $\frac{dr_B}{\sin e_B}$.

An outline of a procedure suitable for evaluating the integrals (77) and (78) will be given in §15.

14. Scattering from a Pulse-Like Primary Wave in a Spherically Symmetrical Model

In this case the mean square amplitude of scattered waves at P is given by substituting from equation (62) into equation (76). We thus obtain

$$\langle\theta^2\rangle \doteq \iiint\limits_{V_1} J\left(g_1X^4 + g_2X^2 + g_3\right)\exp(-\,2X^2)\,de_A\,d\psi_B\,dr_B\ , \qquad (80)$$

where

$$J = \frac{f_C I\sigma^3}{4\sqrt{\pi}\lambda^4 Y^{5/2}}\ \frac{\cos e_A\ \cot e_B'}{r_C{}^2\eta_B\ \sin e_B\ \sin e_C\ \sin \Delta_{BC}}\left|\frac{d^2T_{BC}}{d\Delta^2{}_{BC}}\right|\ , \qquad (81)$$

and where X is now given by

$$X = \frac{\alpha_0}{\lambda}\left(T_{AB} + T_{BC} - t\right)\ . \qquad (82)$$

In general, if for a particular observation point C and a particular time t under consideration, the associated values of the scattering angle ϕ are small to moderate, then it happens that X varies only relatively slowly throughout the region V. In such circumstances the integrand of (80) would therefore vary only slowly throughout

V so the integral can be evaluated numerically using the same procedure as for the integrals (77) and (78) (to be outlined in §15).

For larger values of ϕ, however, X will change more rapidly and the main contribution to the integral will come from that limited part of V where $X^2 < 1$. Under these circumstances it is advantageous to integrate over X analytically as follows.

We first change the integration variables in (80) from (e_A, ψ_B, r_B) to (e_A, ψ_B, s_B) where s_B denotes the length of the ray from A to B. The expression (80) then becomes

$$<\theta^2> = \iiint J\left(g_1X^4 + g_2X^2 + g_3\right)\exp(-2X^2)\sin e_B de_A d\psi_B ds_B \ . \qquad (82)$$

Now let O_1 denote a point on the ray defined by (e_A, ψ_B) at which $\tau' = 0$ (equation (79)), and let O_1xyz denote a local rectangular cartesian coordinate system which has its origin at the point O_1, its x-axis in the direction of the ray passing through O_1 and which contains the field point P in the plane $z = 0$. We shall suppose that the ray through O_1 may be treated as coinciding with the x-axis in

an interval $-a < x < b$, to be specified shortly, and that in the same interval the pencil of rays defined by the ranges $e_A - \frac{de_A}{2}$ to $e_A + \frac{de_A}{2}$ and $\psi_B - \frac{d\psi_B}{2}$ to $\psi_B + \frac{d\psi_B}{2}$ may be treated as parallel. We now replace ds_B by dx in equation (82) and integrate along the elementary pencil from $x = -a$ to $x = +b$. The corresponding contribution to the integral (82) is then given by

$$d\langle\theta^2\rangle = \left\{\int_{x=-a}^{x=b} J\left(g_1X^4 + g_2X^2 + g_3\right)\exp(-2X^2)dx\right\}\sin e_B de_A d\psi_B \ . \qquad (83)$$

Since by choice of O_1 we have $T_{AO_1} + T_{O_1C} - t = 0$, the equation (82) gives, to sufficient accuracy,

$$X = \frac{\alpha_0}{\lambda}\left(\left(T_{AB} - T_{AO_1}\right) + \left(T_{BC} - T_{O_1C}\right)\right),$$

$$= \frac{1}{\lambda}\left(x - (x\cos\phi + y\sin\phi)\right),$$

$$= 2(px + qy)/\lambda \ , \qquad (84)$$

where $p = \sin^2\phi/2$ and $q = -\sin\phi/2\cos\phi/2$ (cf. equation (34)). Changing the integration variable from x to X in equation (83) by use of (84), we obtain

$$d\langle\theta^2\rangle = \left\{\int_{X=-\frac{2pa}{\lambda}}^{X=\frac{2pb}{\lambda}} \frac{\lambda J}{ap}\left(g_1X^4 + g_2X^2 + g_3\right)\exp(-2X^2)dX\right\}\sin e_B de_A d\psi_B \ ,$$

(85)

where we have set $y = 0$ in equation (84) (since the pencil of rays is indefinitely thin). Approximating further by treating $J \sin e_B$, p, g_1, g_2 and g_3 as constant on the interval $-a < x < b$ and integrating (85) with respect to X, we obtain

$$d\langle\theta^2\rangle = \frac{\lambda J}{2p} K(a,b) \sin e_B \, de_A d\psi_B \,, \tag{86}$$

where

$$\begin{aligned} K(a,b) = {} & g_1 \left\{ \frac{3\sqrt{\pi}}{8\times 2^{5/2}} \left[\operatorname{erf}\left(\frac{2^{3/2}pa}{\lambda}\right) + \operatorname{erf}\left(\frac{2^{3/2}pb}{\lambda}\right) \right] \right. \\ & - \frac{3p}{8\lambda} \left[a \exp\left(-\frac{8p^2a^2}{\lambda^2}\right) + b \exp\left(-\frac{8p^2b^2}{\lambda^2}\right) \right] \\ & \left. - \frac{2p^3}{\lambda^3} \left[a^3 \exp\left(-\frac{8p^2a^2}{\lambda^2}\right) + b^3 \exp\left(-\frac{8p^2b^2}{\lambda^2}\right) \right] \right\} \\ & + g_2 \left\{ \frac{\sqrt{\pi}}{4\times 2^{3/2}} \left[\operatorname{erf}\left(\frac{2^{3/2}pa}{\lambda}\right) + \operatorname{erf}\left(\frac{2^{3/2}pb}{\lambda}\right) \right] \right. \\ & \left. - \frac{p}{2\lambda} \left[a \exp\left(-\frac{8p^2a^2}{\lambda^2}\right) + b \exp\left(-\frac{8p^2b^2}{\lambda^2}\right) \right] \right\} \\ & + g_3 \left\{ \frac{\sqrt{\pi}}{2\times 2^{1/2}} \left[\operatorname{erf}\left(\frac{2^{3/2}pa}{\lambda}\right) + \operatorname{erf}\left(\frac{2^{3/2}pb}{\lambda}\right) \right] \right\} , \end{aligned} \tag{87}$$

and where $J \sin e_B$, p, g_1, g_2 and g_3 are to be evaluated at the point O_1.

So far, the range $(-a,b)$ has not been defined. It should first be noted that no matter how large a and b are taken, the contributions to the integral (83) from outside the range $-\lambda/(2\sin^2\phi/2) < x < \lambda/(2\sin^2\phi/2)$ are negligible. We need therefore be concerned with defining a and b only in cases where it is appropriate to take a or b less than $\lambda/(2\sin^2\phi/2)$. Such cases may arise when the region V containing the irregularities is limited in extent (as, for example, where V is a concentric spherical shell $r_1 < r < r_2$). It is then appropriate to take a and b so that $x = -a$ and $x = b$ correspond to the points of entry and exit of the ray (e_A, ψ_B) to and from the region V. Upon combining the contributions (86) from all the ray pencils, equation (80) becomes

$$\langle\theta^2\rangle = \iint_{\Sigma} \frac{\lambda J\ K(a,b)\ \sin e_B\ de_A d\psi_B}{2p}, \tag{88}$$

where in general $a = a(e_A, \psi_B)$ and $b = b(e_A, \psi_B)$ and Σ denotes the domain of (e_A, ψ_B) corresponding to V_1.

If the dimensions of V and the magnitude of ϕ are such that $2pa/\lambda$ and $2pb/\lambda$ exceed unity, then we may approximate by taking $a \to \infty$ and $b \to \infty$. Equation (87) then becomes

$$K(a,b) = \frac{3}{4}\frac{\sqrt{\pi}}{2^{5/2}} g_1 + \frac{1}{2}\frac{\sqrt{\pi}}{2^{3/2}} g_2 + \frac{\sqrt{\pi}}{2^{1/2}} g_3 , \tag{89}$$

which, upon substituting from equations (36), (38), (41) and (63), becomes

$$K(a,b) = \frac{3}{4}\sqrt{\frac{\pi}{2}}\,\Gamma^2 , \tag{90}$$

where (remarkably!) the factor Γ^2 (equation (41)) is the same factor as appears in the scattering formulas for simple harmonic and random primary waves.

If the above conditions are satisfied for all (e_A, ψ_B) inside any domain $\Sigma(e_A, \psi_B)$, then by equations (81), (88) and (90) the scattering from that part of V intercepted by the tube of rays defined by Σ is given by

$$\langle\theta^2\rangle = \frac{3}{32\sqrt{2}} \frac{f_C I}{r_C^{\,2}} \iint_\Sigma \frac{\Gamma^2\sigma^3}{\eta_B p \lambda^3 Y^{5/2}} \frac{\cos e_A}{\sin e_C} \frac{\cot e_B'}{\sin \Delta_{BC}} \left|\frac{d^2 T_{BC}}{d\Delta_{BC}^2}\right| de_A d\psi_B , \tag{91}$$

where the point B is here to be taken as the point on the ray (e_A, ψ_B) where $\tau' = 0$ (equation (79)).

to determine T_{AB}, Δ_{AB} and e_B.

(5) The epicentral distance Δ_{BC} from the scattering point B to the observation point C is then given by

$$\cos \Delta_{BC} = \cos \Delta \cos \Delta_{AB} + \sin \Delta \sin \Delta_{AB} \cos \psi_B \ . \tag{92}$$

(6) Ordinary ray theory is then applied to determine the travel time T_{BC}, the angles of emergence e_B' and e_C and $d^2T_{BC}/d\Delta^2_{BC}$, corresponding to Δ_{BC} and r_B.

(7) The angle χ between the diametral planes containing the primary wave ray AB and the scattered wave ray BC (see figure 3) is given by

$$\cos \chi = \frac{\cos \Delta_{AB} \cos \Delta_{BC} - \cos \Delta}{\sin \Delta_{AB} \sin \Delta_{BC}} \ . \tag{93}$$

(8) The angle ϕ between the direction of the primary wave ray at B and the associated scattered wave ray is given by

$$\cos \phi = \sin e_B \sin e_B' + \cos e_B \cos e_B' \cos \chi \ . \tag{94}$$

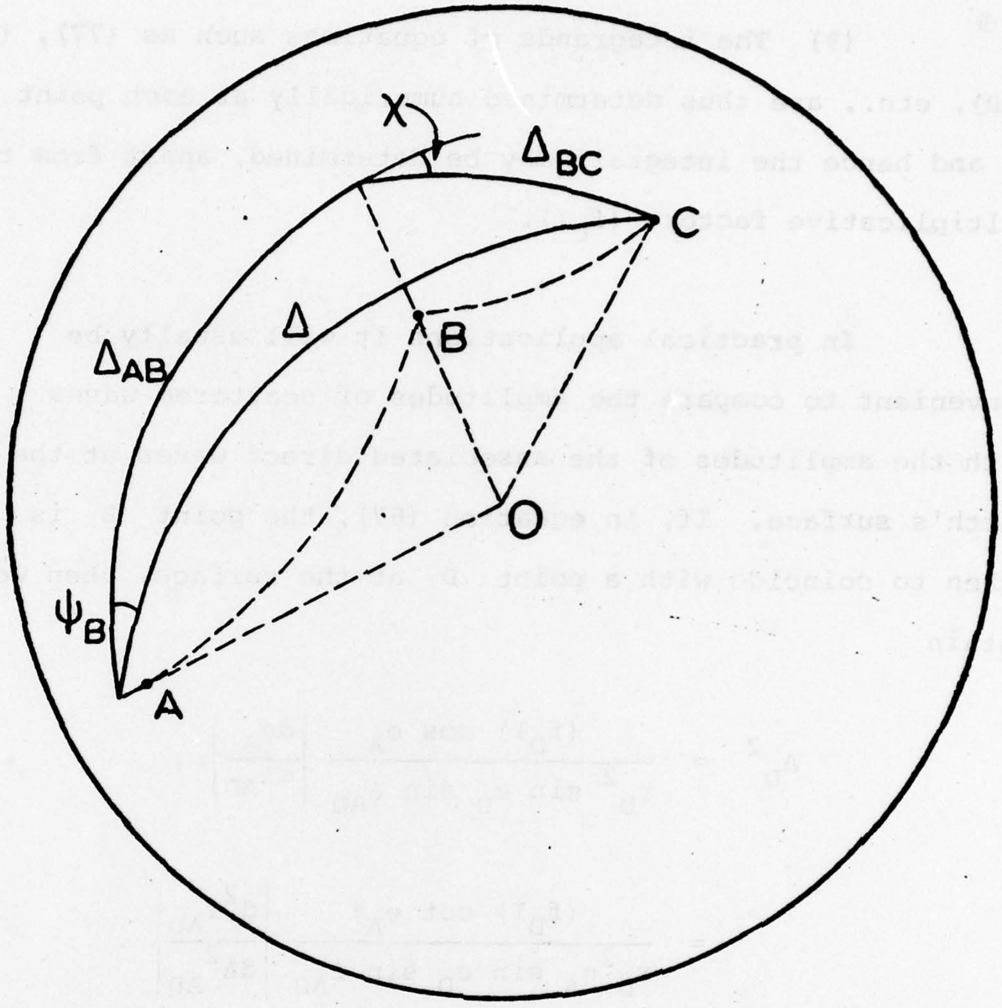

Fig. 3 Diametral planes and angles associated with the source A, the receiver C and a scattering point B.

(9) The integrands of equations such as (77), (78), (80), etc., are thus determined numerically at each point of V and hence the integrals may be determined, apart from the multiplicative factor $(f_C I)$.

In practical applications it will usually be convenient to compare the amplitudes of scattered waves with the amplitudes of the associated direct waves at the Earth's surface. If, in equation (67), the point B is taken to coincide with a point D at the surface, then we obtain

$$A_D^2 = \frac{(f_D I)\cos e_A}{r_D^2 \sin e_D \sin \Delta_{AD}} \left| \frac{de_A}{d\Delta_{AD}} \right| ,$$

$$= \frac{(f_D I)\cot e_A}{r_D^2 \eta_A \sin e_D \sin \Delta_{AD}} \left| \frac{d^2 T_{AD}}{d\Delta^2_{AD}} \right| . \tag{95}$$

Since $f_D = f_C$ the equations for scattered wave amplitudes, and also the equation (95) for direct wave amplitudes all contain the factor $(f_C I)$. The relative amplitudes are therefore independent of this factor.

A further point to be noted is that if either the primary wave or the resulting scattered waves pass through interior boundaries at which discontinuities in ρ_0, k_0 or

μ_0 occur, then the amplitudes should be modified appropriately to take account of energy losses by reflection and conversion from P to S. In some applications allowance for attenuation effects would also need to be made.

16. Concluding Remarks

In deriving the various equations given in this paper, a number of approximations and simplifying assumptions have been made. In each particular application it will therefore generally be necessary to consider whether or not the various approximations made are satisfactory. In general, it is to be expected that the theory will give reliable results whenever the primary wave may be reasonably approximated by a plane wave in each part of the scattering region whose linear dimensions are comparable with the characteristic size of the postulated inhomogeneities. For short period waves this will usually be the case except near caustics and geometrical shadow boundaries. Even in these exceptional cases it is not unreasonable to expect on physical grounds that formulas such as (77), (78) and (91) will provide satisfactory first approximations of scattered wave amplitudes. In this connection it is notable that in applications involving scattering from the vicinity of caustics, the integrands of equations (77) and (78) are not singular in spite of the infinite amplitudes associated with caustics on simple ray theory. In fact, equations (77) and (78) show that on the theory given, the scattering amplitudes associated with any particular ray tube leaving the source

are (almost) independent of variations in the cross-sectional area of the ray tube along its length.

The above theory has been applied to calculate amplitudes of waves scattered from PKP waves in the lowest 200 km of the mantle. The results show that fluctuations in density and elastic parameters of order one per cent in that region fully account for the observed amplitudes of precursors to PKIKP. Although, as indicated above, the results on scattering from the vicinity of caustics need to be treated with some caution, in the case of PKP the numerical calculations give practically the same amplitudes for waves scattered from P <u>before</u> entering the core as for waves scattered from PKP <u>after</u> leaving the core. The plane wave approximation used in this paper should be fully satisfactory in the former case at least, which is sufficient to establish the plausibility of the scattering mechanism. Further details on application of the theory to the PKIKP precursor problem will be published in a separate paper.

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